Study of Compactons and Solitons using Finite Element Method

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In this research, we study some properties of compactons using Finite Element Method (FEM). This method is complicated for programming and very time consuming; but it is an accurate method. Using this method, we studied soliton properties and obtained results were acceptable. Then we studied compactons; Compactons are solitons with finite width or on the other hand solitons with no tail. This defined property for compactons was not observed in our simulation after a long period of time. It seems that breaking of compacton occurred regarding the entity of compacton equations, not by numerical error. In compactons collision, particle-like manner was not observed at all during this research. Perhaps it is due to suddenly vanishing of compactons on both ends.

Introduction

Solitary waves were observed in 1834 for the first time. These waves have constant shape across time. It happens because of the balanced simultaneous effect of nonlinear and dispersive terms. Nonlinear term reduces the width of the wave shape and dispersive term makes it wide. Soliton is a solitary wave. One of the equations which have soliton solution is KdV equation. KdV is a special case of general partial differential equation k(m, n) [1],

$$u_t + (u^m)_x + (u^n)_{xxx} = 0 \qquad , m > 0\,,\ \ 1 < n \leq 3 \tag{1}$$

This equation have compacton solutions for special values of m and n, e. g. $m = n = 2$ or $m = n = 3$. Compactons, by definition, have some characteristic properties of solitons such as particle-like elastic collision [2]. The shape of these waves remains unchanged after collision. Compctons have some basic differences with solitons too, such as

1- Compactons, unlike solitons, have finite width [2].
2- Traveling velocity of compactons, unlike solitons, is independent of width [2].

Due to suddenly vanishing shape of compactons on both ends, numerical study of these equations is difficult. Some numerical methods have been used for solving k(m, n) before, e. g. Pseudo Spectral Method [1], Discontinuous Galerkin Method [3,4] and Finite Difference Method [5]. It seems that some published results have been the authors' expectations, not real numerical solution! This paper has been extracted from more than hundreds hours simulation of k (m, n) equation with Finite Element Method by MATLAB. This method is very complicated for programming and simulation.

Solitons

Some solutions of nonlinear KdV equations are solitons. KdV equation has been written as:

$$u_t - 6uu_x + u_{xxx} = 0 \tag{2}$$

t and x indices are time and space derivatives respectively. Balanced effect of nonlinear term uu_x and dispersive term u_{xxx} , cause initial wave shape to remain unchanged. Solution of KdV equation is a traveling wave with general form $u(x,t) = f(x - ct)$ and c is a constant showing wave velocity. Soliton solution of KdV equation is

$$u(x,t) = -\frac{1}{2}c.sech^2(x - ct) \tag{3}$$

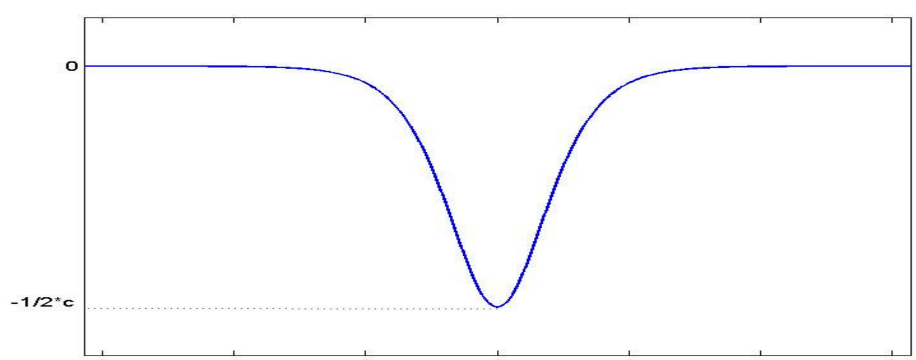

Figure 1. Soliton solution of KdV equation

Compactons

k(m, n) equations were introduced for studying the role of dispersion in the waves. General form of these equations is very similar to KdV

$$u_t + (u^m)_x + (u^n)_{xxx} = 0 \qquad , m > 0 \, , 1 < n \leq 3 \tag{4}$$

and t, x indices are time and space derivatives respectively. As a special case, k(m, n) equation for $m = 2$ and $n = 1$ is KdV equation. Characteristic property of solution of these equations is completely particle-like elastic collision. Unlike solitary waves with infinite width, these solutions have finite widths or on the other hand they have no tail [2]; so, they are compact and called compacton. In some articles, these equations were investigated for special values of m and n, and compacton solution was extracted, e. g. for $m = n = 2$ and $m = n = 3$.

k(2, 2) equation is written as

$$u_t + (u^2)_x + (u^2)_{xxx} = 0 \tag{5}$$

and have closed form solution

$$u_c(x,t) = \begin{cases} \frac{4c}{3} cos^2\left(\frac{x-ct}{4}\right), & |x-ct| \leq 2\pi \\ 0 \, , & otherwise \end{cases} \tag{6}$$

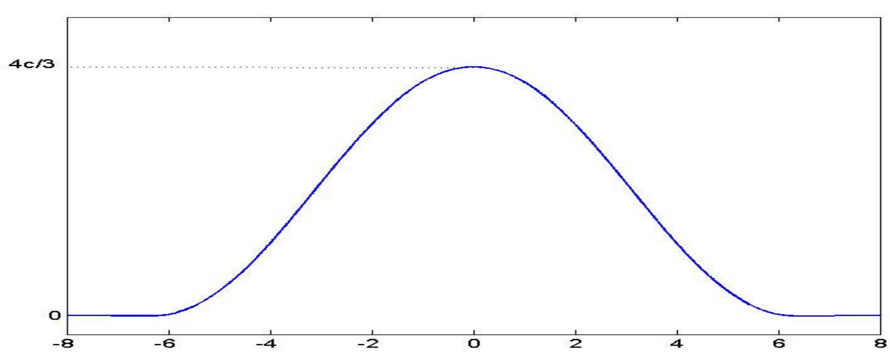

Figure 2. k(2, 2) solution

k(2, 2) solution is invariant under $\begin{cases} u \to -u \\ t \to -t \end{cases}$; so, under this transformation it indicates an anti-compacton, traveling in opposite direction [1]. k(3, 3) equation is written as

$$u_t + (u^3)_x + (u^3)_{xxx} = 0 \qquad (7)$$

and has closed form solution [1]

$$u_c(x,t) = \begin{cases} \pm\sqrt{\frac{3c}{2}} cos \left(\frac{x-ct}{3}\right), & |x-ct| \le \frac{3\pi}{2} \\ 0\,, & otherwise \end{cases} \qquad (8)$$

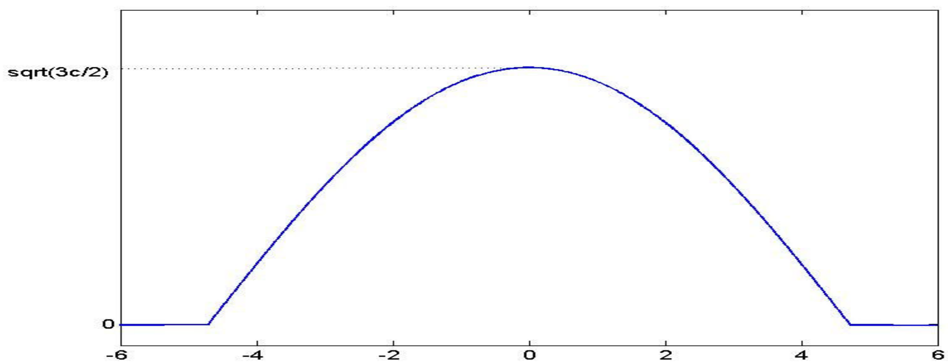

Figure 3. k(3, 3) solution

Finite Element Method

Finite Element Method is a powerful and precise method but it is difficult and time consuming method for solving wide range of ordinary and partial differential equations. It works for both initial and specially boundary value problems. In this method we divide space of problem into subspaces with the same sizes or, in most cases, different sizes. Then in each subspace, the solution of differential equation is approximated by the series of some arbitrary basic functions with unknown coefficients. We should try to find these unknown coefficients and consequently the solution. Compactons were studied by different methods such as Implicit Finite Difference Method, Explicit Finite Difference Method, etc. These methods are neither reliable nor stable for compacton equations. Smaller sizes of steps do not have desirable effect on stability of solution. In implicit method, a forth order term is added to the equation as an absorber. But in Finite Element Method, the solutions are more reliable and more similar to closed form solution.

Conclusion

In one part of our research, we simulate the KdV equation. It obtained soliton travelling with constant shape (Figure 4), collision, and then separation (Figure 5).

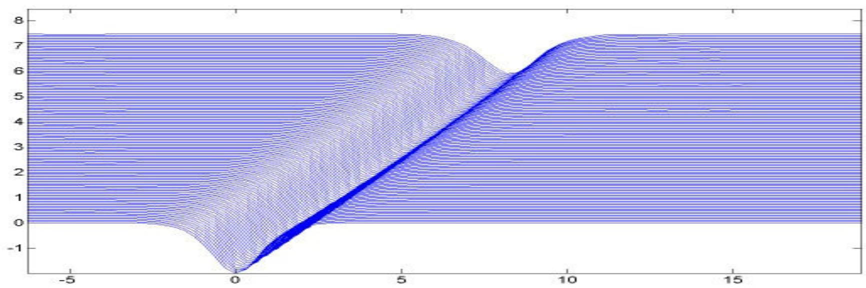

Figure 4: Soliton moves without change in shape

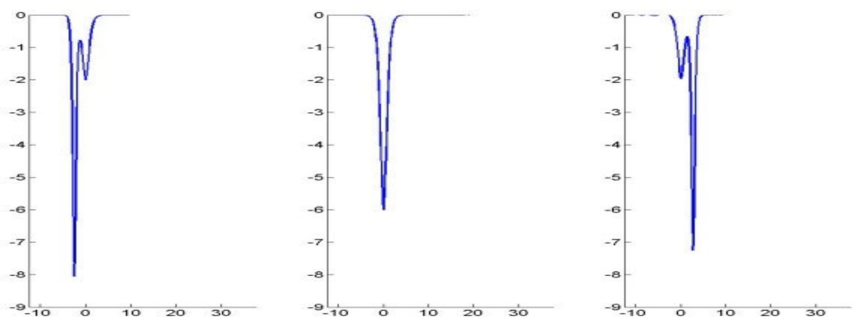

Figure 5: Collision of two solitons and separation

We saw that if we solve KdV equation with arbitrary initial wave shape, some perturbations leave the shape and soliton solution is appeared and travel without change (Figure 6). For comparing between extracted soliton from arbitrary initial shape and closed form solution of KdV, we insert equation (1) by dots in figure (6) at related time.

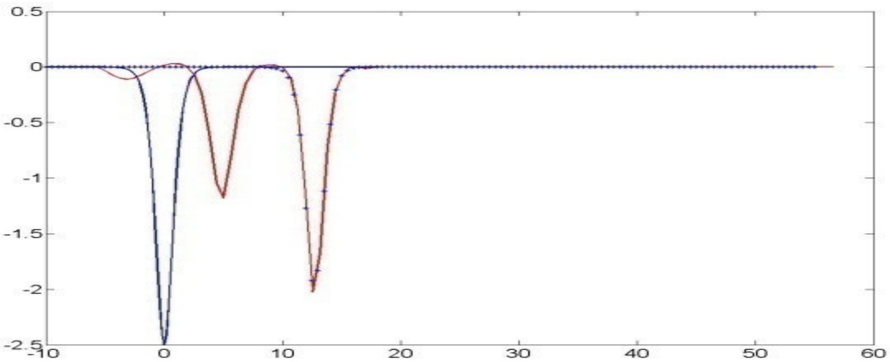

Figure 6: Some perturbations leave initial arbitrary shape (blue) and the soliton moves without change in shape (red shape in right), dots on right valley show the soliton shape coincidence.

But for the compactons, even with closed form solution of k(2, 2) and k(3, 3) as initial wave shapes, in the time evolution of related equation, some perturbations appear and then blow up. We investigated k(2, 2) and k(3, 3) by Finite Element Method with a wide variety of basic functions, space step sizes and time step sizes. This happens even with reducing space and time step sizes for long time simulation for all basic functions (Figure 7,8).

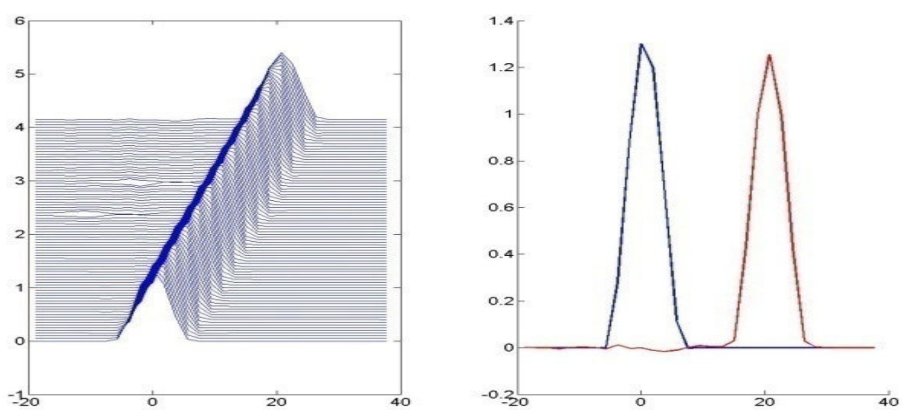

Figure 7: Movement of k(2, 2) compacton before crashing

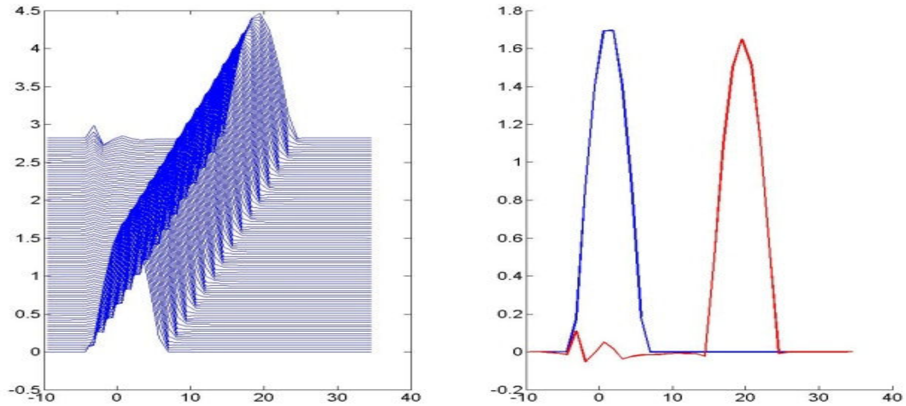

Figure 8: Movement of k(3, 3) compacton before crashing

Not only in Finite Element Method, but also for all frequently used numerical methods, the compacton has not time evolution and particle-like collision. Perhaps, main role in divergence is caused by suddenly vanishing of the compactons on both ends. This event causes discontinuity in derivatives. Is it true that the main part or all of divergence is caused by numerical method? We should answer to this question carefully. All of numerical methods have some round off or truncation errors. But it is accepted that numerical methods are applicable, specifically for the problems with no analytic or closed form solution. One of the most accurate numerical methods is Finite Element Method and we obtained some acceptable results for solitons by this method. So, perhaps some of properties that enumerated for compactons are unreal. Can we confine a wave in this limit and relate particle like manner to it?

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